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NOTE

DISTANCE REGULAR GRAPHS OF DIAMETER 3 AND STRONGLY REGULAR GRAPHS

A.E. BROUWER

Mathematisch Centrum, Kruislaan 413, 1098 SJ Amsterdam, The Netherlands

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In [1] N.L. Biggs mentions two parameter sets for distance regular graphs that are antipodal covers of a complete graph, for which existence of a corresponding graph was unknown. Here we settle both cases by proving that one does not exist, while there are exactly two nonisomorphic solutions to the other. We note some relations with strongly regular graphs and generalized quadrangles.

1. Merging classes in an association scheme

Suppose $(X, \{I, R_1, R_2, R_3\})$ is an association scheme with 3 classes. It often happens that for suitable indexing we have that $(X, \{I, R_1 \cup R_2, R_3\})$ is an association scheme with 2 classes. In fact this happens precisely when

$$p^1_{11}+p^1_{12}+p^1_{21}+p^1_{22}=p^2_{11}+p^2_{12}+p^2_{21}+p^2_{22},$$

or, equivalently, when $p^1_{33}=p^2_{33}$, or, equivalently, when R_3 has only three distinct eigenvalues.

(Indeed, the first condition says that $(X, R_1 \cup R_2)$ is a strongly regular graph, while the latter two conditions each say that (X, R_3) is strongly regular.)

For distance regular graphs with intersection array $(k, b_1, b_2; 1, c_2, c_3)$ the first condition reduces to $k=b_2+c_3-1$. (Note that the Hoffman bound for cliques in a strongly regular graph $(X, R_1 \cup R_2)$ is $k+1$ so that one gets many maximal cliques, namely the 'stars' $x^\perp$ for $x \in X$. In fact each point outside a star has $c_3=k-b_2+1$ neighbours in the star, so stars are regular cliques in the sense of Neumaier [5].)

Similarly, merging R_1 and R_3 in such a graph produces a strongly regular graph precisely when R_2 has only three distinct eigenvalues, which happens when

$$c_3(a_3+a_2-a_1)=b_1a_2$$

(where $k=a_i+b_i+c_i$ $(i=1,2,3)$, $b_3=0$, $c_1=1$).

Of course merging R_2 and R_3 cannot work, since the matrices R_2 and R_3 are linear combinations of powers of R_1, so that if R_1 has only three eigenvalues then the whole scheme has only three eigenspaces, which is impossible.

2. Spreads in generalized quadrangles

Reversing the process of the previous section, we construct some distance regular graphs of diameter 3 by 'unmerging' certain strongly regular graphs.

Suppose $\mathcal{S}$ is a spread (i.e., a partition of the point set by lines) in a generalized quadrangle $(X, \mathcal{L})$. Define a graph with vertex set X and edges (u, v), where the points u, v are collinear and the line uv does not belong to the spread $\mathcal{S}$. This graph is distance regular with diameter 3. In fact, if the generalized quadrangle has parameters (s, t) with $t > 1$ then this graph has intersection array

$$(st, s(t-1), 1; 1, t-1, st)$$

so that we have an antipodal $(s+1)$-cover of the complete graph K_{st+1}.

More generally, the same construction works when we start with a pseudo-geometric strongly regular graph (with the parameters of GQ(s, t)) which possesses a partition into cliques of size $s+1$. Conversely, any graph with the above intersection array is obtained in this way. (This is a special case of the situation considered in the previous section.)

3. Biggs' open cases

In [1] Biggs mentions two parameter sets for possible antipodal covers of a complete graph as open.

The first has intersection array $(8, 6, 1; 1, 3, 8)$ and is, as we saw above, derived from a strongly regular graph with parameters $(v, k, \lambda, \mu) = (27, 10, 1, 5)$. But there is a unique such graph, the point graph of the unique GQ$(2, 4)$ [the well known "27 lines on a cubic surface"], and this graph possesses exactly two nonisomorphic spreads (see Brouwer & Wilbrink [4]). Thus, there exist precisely two nonisomorphic graphs with this intersection array.

The second open case had intersection array $(8, 4, 1; 1, 1, 8)$. Now the fact that $c_2 = 1$ means (as was already observed in Bose & Dowling [2]) that the graph induced on the neighbours of a fixed point is a union of cliques—in this case two K_4's—so that one has a geometry with points and lines with girth at least five. In our case we have 45 points, and two 5-lines on each point, so 18 lines altogether. A given line is adjacent to 5 others and has distance two to 20 lines, so that $18 \geq 1 + 5 + 20$, ridiculous. Thus, no such graph exists.

Biggs also mentions the smallest open cases for primitive (i.e., not bipartite and not antipodal) distance regular graphs of diameter 3. Four of these six cases give

Table 1

$(k, b_1, b_2; 1, c_2, c_3)$	Eigenvalues	Merge	(v, k, λ, μ)	Comments
(7, 6, 6; 1, 1, 2)	$7, 3^{66}, -1^{77}, -4^{32}$	$R_1 \cup R_2$	(176, 49, 12, 14)	One srg known [3], does not give 3-class scheme
(8, 7, 5; 1, 1, 4)	$8, 3^{54}, -1^{50}, -4^{30}$	$R_1 \cup R_2$	(135, 64, 28, 32)	One srg known $(O_8^+(2))$, does not work either
(11, 10, 4; 1, 1, 5)	$11, 4^{55}, 1^{77}, -4^{77}$	$R_1 \cup R_3$	(210, 99, 48, 45)	No srg known
(13, 10, 7; 1, 2, 7)	$13, 5^{39}, -1^{78}, -5^{26}$	$R_1 \cup R_2$	(144, 78, 42, 42)	Many nonisomorphic srg's known; not checked for 3-class schemes

rise to a strongly regular graph (as in section 1). The parameters are as displayed in Table 1.

References

[1] N.L. Biggs, Distance regular graphs with diameter three, Annals Discr. Math. 15 (1982) 69–80.

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[3] A.E. Brouwer, Polarities of G. Higman's symmetric design and a strongly regular graph on 176 vertices, Aequationes Math. 25 (1982) 77–82.

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[5] A. Neumaier, Regular cliques in graphs and special $1\frac{1}{2}$-designs, in: Finite Geometries and Designs, LMS Lecture Note Series 49 (Cambridge Univ. Press, 1981) 244–259.